Grade 7/8 Math Circles

March 25th - 28th, 2024

Continued Fractions - Problem Set

1. Rewrite each rational number as the unique $\frac{a}{b}$ representation where a and b are both integers.

   (a) $\dfrac{1/2}{2}$

   (b) $\dfrac{5/7}{8/4}$

   (c) $\dfrac{4.3}{8.5}$

   (d) $\dfrac{2.3}{3.2}$

   Solution:

   (a) $\dfrac{1/2}{2} = \dfrac{1}{4}$

   (b) $\dfrac{5/7}{8/4} = \dfrac{20}{56}$

   (c) $\dfrac{4.3}{8.5} = \dfrac{43/10}{85/10} = \dfrac{430}{850}$

   (d) $\dfrac{2.3}{3.2} = \dfrac{23/10}{32/10} = \dfrac{230}{320}$

2. A little monkey had 60 peaches.

   On the **first** day he decided to keep $\frac{3}{4}$ of his peaches. He gave the rest away. Then he ate one.

   On the **second** day he decided to keep $\frac{7}{11}$ his peaches. He gave the rest away. Then he ate one.

   On the **third** day he decided to keep $\frac{5}{9}$ of his peaches. He gave the rest away. Then he ate one.

   On the **fourth** day he decided to keep $\frac{2}{7}$ of his peaches. He gave the rest away. Then he ate one.

On the **fifth** day he decided to keep $\frac{2}{3}$ of his peaches. He gave the rest away. Then he ate one. How many peaches did the monkey have left at the end?

Solution: After the $1^{\textbf{st}}$ **day**, the little monkey kept $60 \times \frac{3}{4} = 45$ peaches. He then ate one, so he had 44 peaches left.
After the $2^{\textbf{nd}}$ **day**, the little monkey kept $44 \times \frac{7}{11} = 28$ peaches. He then ate one, so he had 27 peaches left.
After the $3^{\textbf{rd}}$ **day**, the little monkey kept $27 \times \frac{5}{9} = 15$ peaches. He then ate one, so he had 14 peaches left.
After the $4^{\textbf{th}}$ **day**, the little monkey kept $14 \times \frac{2}{7} = 4$ peaches. He then ate one, so he had 3 peaches left.
After the $5^{\textbf{th}}$ **day**, the little monkey kept $3 \times \frac{2}{3} = 2$ peaches. He then ate one, so he only had one peach left.
Therefore the little monkey had one peach left

3. Write the following continued fraction expansions in the fraction form. No need to simplify!
   (a) $[1, 2, 4, 5]$
   (b) $[0, 9, 4, 3]$
   (c) $[1, 7, 3, 2]$
   (d) $[4, 7, 2]$

Solution: a. $1 + \cfrac{1}{2 + \cfrac{1}{4 + \frac{1}{5}}}$

b. $\cfrac{1}{9 + \cfrac{1}{4 + \frac{1}{3}}}$

c. $1+\cfrac{1}{7+\cfrac{1}{3+\cfrac{1}{2}}}$

d. $4+\cfrac{1}{7+\cfrac{1}{2}}$

4. Solve for the rational numbers associated to the continued fraction expansions given in Question 3.

Solution:

(a) $[1,2,4,5]=\dfrac{68}{47}$

(b) $[0,9,4,3]=\dfrac{13}{120}$

(c) $[1,7,3,2]=\dfrac{58}{51}$

(d) $[4,7,2]=\dfrac{62}{15}$

5. Solve for the continued fraction expansions of the reciprocals of the rational numbers you solved for in Question 4, what do you notice?
Note: the reciprocal of a rational number $\dfrac{a}{b}$ *is* $\dfrac{b}{a}$.

Solution:

(a) $\dfrac{47}{68}=[0,1,2,4,5]$

(b) $\dfrac{120}{13}=[9,4,3]$

(c) $\dfrac{51}{58}=[0,1,7,3,2]$

(d) $\frac{15}{62} = [0, 4, 7, 2]$

What we notice is that if the reciprocal of the fraction is less than the original fraction we add a 0 to the beginning of its continued fraction expansion and in the case where the reciprocal of the fraction is greater than the original fraction then we take away the 0 at the beginning of its continued fraction expansion.

6. Solve for the continued fraction expansions of the following rational numbers:

   (a) $\frac{49}{11}$

   (b) $\frac{423}{95}$

Solution:

(a) $\frac{49}{11} = [4,\ 2,\ 5]$

(b) $\frac{423}{95} = [4,\ 2,\ 4,\ 1,\ 3,\ 2]$

7. Solve for the irrational number associated with the following infinite continued fraction expansions.

   (a) $[3, 2, 3, ...] = [3, \bar{2}]$

   (b) $[1, 4, 1, ...] = [1, \bar{4}]$

Solution:

(a) the irrational number associated with the following infinite continued fraction expansion $[3, 2, 3, ...] = [3, \bar{2}]$ is $\frac{3 + \sqrt{15}}{2}$

(b) the irrational number associated with the following infinite continued fraction expansion $[1, 4, 1, ...] = [1, \bar{4}]$ is $\frac{1 + \sqrt{2}}{2}$

8. Solve for the infinite continued fraction expansions of the following irrational numbers (try finding the pattern as early as possible for fun!).

(a) $\sqrt{3}$

(b) $\sqrt{5}$

Solution:

(a) the infinite continued fraction expansions of $\sqrt{3}$ is $[1, 1, 2, 1, 2, ...]$

(b) the infinite continued fraction expansions of $\sqrt{5}$ is $[2, 4, 4, 4, ..]$